Def: A set A is <u>countable</u> if it is either denumerable ($|A| = \aleph_0$) or finite ($|A| = n$ for some $n \in \mathbb{N}$).

(~~Last~~ In <u>notes</u>: • If A is denum. and $f: B \to A$ injective, then B is countable. ($|B| \leq |A|$)

• If A is denum. and $f: A \to B$ surjective then B is countable.)

<u>Today</u>: If A and B are denumerable sets, then $A \times B$ is denumerable.

<u>Ex</u>: Let's show $\mathbb{N} \times \mathbb{N} = \{(a,b) : a \in \mathbb{N}, b \in \mathbb{N}\}$ is denumerable.

We can list the elements of $\mathbb{N} \times \mathbb{N}$:

$$\underbrace{(1,1)}_{\text{1st diagonal}}, \underbrace{(2,1), (1,2)}_{\text{2nd diagonal}}, \underbrace{(3,1), (2,2), (1,3)}_{\text{3rd diagonal}}, \underbrace{(4,1), (3,2), (2,3) \ldots}_{\text{4th diagonal.}}$$

Consider the function $g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$$g(a,b) = \frac{(a+b)(a+b-1)}{2} - (a-1).$$

$g(1,1) = \frac{2 \cdot 1}{2} - (1-1) = 1$

$g(2,1) =$ — — — $= 2$

$g(1,2) =$ — — — — $= 3$

$g(3,1) =$ — — — — — $= 4$

$\vdots$ $\qquad\qquad$ $\vdots$

<u>Prop:</u> The function g is both injective and surjective.

<u>Comment:</u> g is the inverse of the function on the previous page.

<u>Ex:</u> • $\mathbb{Z} \times \mathbb{N}$ is denumerable.

<u>Proof:</u> We have shown there's a bijection

$$g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

And there's a bijection $h : \mathbb{Z} \longrightarrow \mathbb{N}$

$$\begin{matrix} \mathbb{Z} \\ \times \\ \mathbb{N} \end{matrix} \xrightarrow[\mathrm{id}_{\mathbb{N}}]{h \times} \begin{matrix} \mathbb{N} \\ \times \\ \mathbb{N} \end{matrix} \xrightarrow{g} \mathbb{N}$$

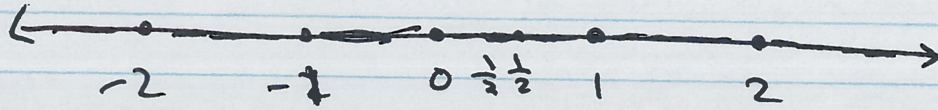
The composition shown is bijective. ∎

Rational numbers:

$$\mathbb{Q} = \left\{ \frac{a}{b} : \begin{array}{l} a \in \mathbb{Z} \\ b \in \mathbb{N} \end{array} \right\}.$$

there's redundancy

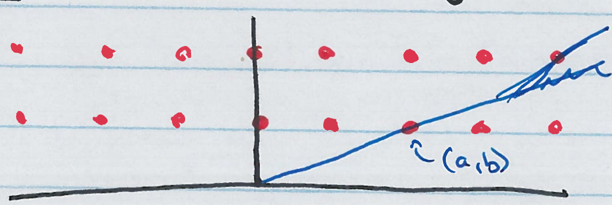
Ex: $\frac{8}{6} = \frac{4}{3}$



Prop: The set $\mathbb{Q}$ is denumerable.

Idea: There's a surjection $\underset{\text{denumerable}}{\mathbb{Z} \times \mathbb{N}} \longrightarrow \mathbb{Q}$

$$(a,b) \longmapsto \frac{a}{b}$$



So $|\mathbb{Z} \times \mathbb{N}| \geq |\mathbb{Q}|$.

Pf: The set $\mathbb{Z} \times \mathbb{N}$ is denumerable.

And $f : \mathbb{Z} \times \mathbb{N} \longrightarrow \mathbb{Q}$

$f(a,b) = \frac{a}{b}$ is surjective.

So $\mathbb{Q}$ is countable. It's not finite, so it's denumerable. ∎

...	$\frac{-2}{2}$	$\frac{-1}{2}$	$\cancel{\frac{0}{2}}$	$\frac{1}{2}$	$\cancel{\frac{2}{2}}$	...
...	$\frac{-2}{1}$	$\frac{-1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	...

Exercises: (1) • Prove $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is denumerable.

(2) • Consider the function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$$f(a,b) = 2^{a-1}(2b-1).$$

Prove f is bijective. (Can you come up with more functions?)

(1) Let $g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be any bijection. Then the composition

$$(\mathbb{N} \times \mathbb{N}) \times \mathbb{N} \xrightarrow{g \times id_{\mathbb{N}}} \mathbb{N} \times \mathbb{N} \xrightarrow{g} \mathbb{N}$$

is a composition of two bijections, therefore bijective.

2) Comment: This is due to the fact that any natural number n can be written uniquely as (power of 2) · (odd number).

Proof: We must prove f is injective and surjective.

Injective: Suppose $f(a_1,b_1) = f(a_2,b_2)$. Then

$$2^{a_1-1}(2b_1-1) = 2^{a_2-1}(2b_2-1)$$

~~Suppose for a contradiction that $a_1 \neq a_2$~~

WLOG, $a_1 \leq a_2$. Then $2b_1 - 1 = 2^{a_2-a_1}(2b_2-1)$.
Since $2b_1-1$ is odd, it follows $2^{a_2-a_1}(2b_2-1)$ is odd. So $\underline{a_1 = a_2}$. Then $2b_1-1 = 2b_2-1$ and so $\underline{b_1 = b_2}$.

Surjective: We must prove that $\forall n \in \mathbb{N}$, $\exists (a,b) \in \mathbb{N} \times \mathbb{N}$, $f(a,b) = n$.
We proceed by induction on n.

Base Case: $n=1$. Then $f(1,1) = 1$.

Inductive step: Suppose that $1, 2, \ldots, n-1$ are all in the image of f.

- If n is odd, then $n = 2b-1$ for some $b \in \mathbb{N}$. Then $f(1,b) = n$ and we are done.
- If n is not odd, then $n = 2m$ for some $m \in \mathbb{N}$. By the inductive hypothesis, there exist $a, b \in \mathbb{N}$ such that $f(a,b) = m$. Then $f(a+1, b) = n$, and we are done. ∎

Another function: Any $n \in \mathbb{N}$ can be uniquely written as
$n = $ (power of 3) $\cdot$ (number not divisible by 3).

So, let

$$g : \mathbb{N} \longrightarrow \{\text{powers of } 3\}$$
$$g(a) = 3^{a-1}$$

$$h : \mathbb{N} \longrightarrow \{\text{natural \#s not a multiple of } 3\}$$
$$h(b) = \frac{6b - 3 + (-1)^{b-1}}{4} = \begin{cases} \frac{3b-1}{2} & b \text{ odd} \\ \frac{3b-2}{2} & b \text{ even} \end{cases}$$

Then

$$f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$
$$f(a,b) = g(a) \cdot h(b) = 3^{a-1}\left(\frac{6b - 3 + (-1)^{b-1}}{4}\right)$$ is bijective.